

Coupling “Classical” and Quantum Variables

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Abstract

Experimentally, certain degrees of freedom may appear classical because their quantum fluctuations are smaller than the experimental error associated with measuring them. An approximation to a fully quantum theory is described in which the self-interference of such “quasiclassical” variables is neglected so that they behave classically when not coupled to other quantum variables. Coupling to quantum variables can lead to evolution in which quasiclassical variables do not have definite values, but values which are correlated to the state of the quantum variables. A mathematical description implementing this backreaction of the quantum variables on the quasiclassical variables is critically discussed.

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It is an observation of long standing that the world around us is (or appears to be) largely classical. The fundamental description of the world is however quantum mechanical. A natural and important question is whether one can formulate an approximate description in which certain degrees of freedom are treated as essentially classical while coupling them to other degrees of freedom which are fully quantum. Such a description might be especially important in exploring the domain between the fully classical and fully quantum regimes. As well, it would be particularly useful in a subject like quantum gravity where the full quantum theory is not known, and one cannot make use of the semiclassical approximation. In both these cases, a problem of particular interest is how one can describe and quantify the backreaction of the quantum variables on the classical ones. The positive and negative features are discussed here of a proposal[1] which gives a mathematical prescription for coupling (quasi)classical and quantum variables with physically desirable behavior.

The traditional approach to coupling classical and quantum variables is to use expectation values wherever quantum variables appear in a mixed set of equations of motion[2]. This treats the full system as essentially classical and has the virtue of producing the realist-desired description of a definite classical evolution. This approach can be criticized on a number of grounds. In particular, an expectation value is not the outcome of a single measurement but is an average of the outcomes of an ensemble of identically prepared measurements. One might have expected that the interaction with the classical variables was in some sense measuring the quantum variables, but it is certainly not averaging over repeated identical measurements[3]. The result of this malapropos usage of expectation value is that this procedure gives physically wrong results when the expectation value deviates from the most likely outcome(s) of a single measurement, as it does for example in bimodal distributions.

An explicit example (cf. [2]) illustrating the difficulty is given by coupling the momentum p_a of a particle-a with the momentum p_b of a second particle-b through the interaction Hamiltonian $H_I = cp_ap_b$. Consider first the fully quantum system, neglecting the self-Hamiltonians of particle-a and -b. Suppose at time $t = 0$ that the position of particle-a is localized in a wavepacket $|\phi(x_a), 0\rangle$ with expected position x_0 and expected momentum zero. Suppose also that at time $t = 0$ particle-b is in a superposition of two

momentum eigenstates of equal and opposite momentum \bar{p}_b

$$|\psi, 0\rangle = \frac{1}{2^{1/2}}(|\bar{p}_b, 0\rangle + |-\bar{p}_b, 0\rangle).$$

(This argument could be made with wavepackets for particle-b, but it is easier to be explicit using eigenstates.) A system initially prepared in a product state $|\phi, 0\rangle|\psi, 0\rangle$ will evolve to a correlated superposition

$$e^{-iH_I t}|\phi, 0\rangle|\psi, 0\rangle = \frac{1}{2^{1/2}}(|\phi(x_a - c\bar{p}_b t), t\rangle|\bar{p}_b, t\rangle + |\phi(x_a + c\bar{p}_b t), t\rangle|-\bar{p}_b, t\rangle). \quad (1)$$

So, for example, if $\phi(x_a) = \pi^{-1/4} \exp(-(x_a - x_0)^2/2)$, then $\phi(x_a - c\bar{p}_b t) = \pi^{-1/4} \exp(-(x_a - c\bar{p}_b - x_0)^2/2)$ is localized about $x_a = x_0 + c\bar{p}_b t$, as one would expect from the solution of the Heisenberg equation of motion.

On the other hand, if particle-a were classical and one coupled its position to the expectation value of the momentum of particle-b, there would be no effect because

$$\langle\psi, 0|p_b|\psi, 0\rangle = 0.$$

This expectation value is the average of the two likely outcomes \bar{p}_b and $-\bar{p}_b$ of a measurement. It is not itself the outcome of any measurement. The classical particle is coupled to a phantom. (The situation would be more dramatic if the states were set up so that the expectation value of p_b in state $|\psi, 0\rangle$ were nonzero.)

A further difficulty is exposed if one imagines that a momentum measurement is subsequently made on $|\psi, t\rangle$ and particle-b is projected into an eigenstate of definite momentum. The expectation value of p_b is suddenly nonzero and the classical particle begins to feel the effect of the coupling. This is very peculiar behavior and would raise the relevance of the question of when a measurement is completed to a daunting level—it would have physically meaningful consequences because the coupling between classical and quantum systems would be changed by the act of measurement.

These defects of coupling to expectation values are commonly interpreted as evidence demonstrating the impossibility of coupling classical and quantum variables. This conclusion is too strong, but nevertheless the example carries an important lesson about the nature of classical-quantum interactions. Consider what would happen if particle-a were made increasingly classical starting from the fully quantum result. The state $|\phi, 0\rangle$ would go

over into a “state” $|(x_0, 0), 0\rangle$ with position $x_a = x_0$ and momentum $k_a = 0$. The result of evolution following from the classical limit of (1) is

$$\frac{1}{2^{1/2}}(|(x_0 + c\bar{p}_b t, 0), t\rangle|\bar{p}_b, t\rangle + |(x_0 - c\bar{p}_b t, 0), t\rangle|-\bar{p}_b, t\rangle). \quad (2)$$

This has a “classical” particle in correlation with the state of a quantum subsystem. The “classical” particle-a does not have a definite position. Its specific location depends on the quantum state. In this example that would not be determined until the position of particle-a were observed or a momentum measurement was made on particle-b. Such measurements would show the position of particle-a to be correlated to the outcome of the momentum measurement of particle-b as common sense would suggest. An important and physically desirable feature of coupling classical and quantum variables then is that it be possible for the value of a “classical” variable to depend on the quantum state to which it is correlated. Such a variable is not classical in the realist sense of always having a definite value, so to distinguish this, it shall be called *quasiclassical*.

One may well ask in what sense a variable is to be classical if it does not take definite values. The answer lies at the heart of the new proposal. A quasiclassical variable is one whose self-interference effects can be neglected. It is classical because it does not exhibit observable interference phenomenon in its self-interaction. When coupled to a quantum system, the correlation with quantum states will generally induce interference behavior on the quasiclassical variables, but it is not an intrinsic property of those variables. A mathematical encoding of this definition will be proposed below, but it is valuable to elaborate on its intuitive meaning first.

Every experiment has a scale of resolution or minimum experimental error with which a measurement can be made. A quasiclassical variable is one whose quantum fluctuations are negligible (or at least small) compared to the experimental error with which the variable is known. This is essentially an operational definition of what it means to appear classical. No variable is actually classical; if examined closely enough, it will be seen to have quantum fluctuations. But if the experimental error is sufficiently large and the wavepacket not too delocalized, the quantum fluctuations will essentially all take place within the error range where they are indistinguishable from (classical) measurement uncertainty. In that instance, the variable is operationally indistinguishable from being classical. It is a stronger assumption

that this condition persist under evolution, but that is the property we desire of classical variables and hence require of quasiclassical ones.

It should be emphasized that the apparent classical nature of a variable is an experimental artifact. Consider the location of the center of mass of a macromolecule of some extended size. The center of mass is not a quasiclassical variable in and of itself simply because the mass is large. Rather it is (if it is) because experiment fails to measure the location of the center of mass to the necessary resolution to see quantum effects. Arguably it is easier to measure the location of a concentrated point-like object of a given mass than to measure the location of the center of mass of a complicated extended object of the same mass. It may be that the extended size and complex geometry of the macromolecule makes identifying the location of the precise center of mass difficult. This is an important remark because mathematically the center of mass variable behaves like a point particle, but experimentally it is not observed as such. Practically speaking, one is satisfied with knowing the macromolecule as a whole is “there,” and the location of the molecule as seen in some averaged sense is happily attributed to be that of the center of mass for theoretical purposes. The motion of the molecule then behaves classically because of the relatively imprecise limits that can be put on its position and momentum. Similar remarks would also hold for the other large scale descriptors of the molecule like its linear dimensions, angular momenta, etc.

The central argument that is exploited to understand the interaction of quantum variables and quasiclassical ones is the following. Quasiclassical variables, as actually part of a fully quantum system, are coupled to other quantum variables. This coupling can produce evolution which extends the wavepacket of a quasiclassical variable beyond the range of its associated experimental error. When this happens, the quasiclassical variable is in correlation with the state of those other variables. If the coupling to the other quantum variables were turned off, the quasiclassical variable would be in a delocalized state which could be binned into a set of experimental error intervals. Within each such interval the quasiclassical state would be persistent by the assumption of negligible self-interference. It is thus operationally classical within each interval. Which particular interval occurs, or which set of intervals is possible, depends on the quantum state to which the quasiclassical variable is correlated. As the knowledge of this state is refined by measurement-observation, knowledge of the quasiclassical variable is also

refined.

One could preemptively observe the quasiclassical variable. Repeated measurements of identically prepared situations would reveal that it does not have the realist property of having a definite value (within experimental error). This is expected: when correlated to other quantum states, a quasiclassical variable need not be localized within a single experimental error range. Conventionally, one attributes this not to the underlying quantum nature of the quasiclassical variable, but to the correlated quantum states. These states are viewed as the outcomes of a quantum “event” which triggered the non-classical behavior. The situation is the same as with Schrödinger’s cat. From the fully quantum standpoint, this attribution is a fiction, but in the quasiclassical framework it “explains” why more than one outcome is possible for a classical object. Once the quasiclassical variable is relocalized within a single measurement interval it will persist within a neighborhood of that size until it is disrupted by interaction with further quantum systems.

The paradigmatic example of a quantum event is that of a spin passing through a Stern-Gerlach apparatus, and this will be discussed below. To take a more extreme example to illustrate the significance of measurement scales, consider the case of gravity. Quantum gravitational fluctuations are expected to be important at scales around the Planck length (10^{-33} cm). At length scales of general interest, they are many orders of magnitude smaller than fluctuations of quantum matter variables. Neglecting quantum gravitational corrections to matter processes relative to the contribution of quantum matter fluctuations is generically justifiable. Since quantum gravitational fluctuations are on a much smaller scale than can be seen experimentally, and this condition persists under ordinary evolution, one can ignore the quantum nature of the gravitational field and treat the background of spacetime as quasiclassical.

There is however the possibility of backreaction of the quantum matter fields on the gravitational background. While quantum matter fluctuations are very small on the length scales typically important for classical gravity and their neglect is usually justified, these fluctuations can lead to qualitative changes in classical evolution, possibly by triggering instabilities. This may be particularly important in the early universe. In a different context, quantum fluctuations of a scalar field amplified by inflation have already been proposed as the source of fluctuations in the cosmic microwave background radiation and as seeds for galaxy formation[4].

A thought experiment makes the point sharper and again illustrates the failing of the prescription of coupling to expectation values. Choptuik[5] has recently shown that classically a black hole forms from spherically symmetric collapse of a massless scalar field whose initial configuration is parametrized by a parameter ρ when ρ exceeds a critical value ρ^* . For $\rho < \rho^*$, no black hole forms and the background settles down to flat space as the scalar field disperses. Imagine a wavepacket in ρ of such initial configurations. Choose the wavepacket to be localized so that it extends into the region above ρ^* while the expectation value of ρ is less than ρ^* , $\langle \rho \rangle < \rho^*$. Coupling to the expectation value would lead to the conclusion that no black hole forms. Physical intuition leads one to expect instead that a black hole should form with a probability reflecting the likelihood of finding the scalar field with $\rho > \rho^*$. One would say that quantum fluctuations of the scalar field—reflected by the nonvanishing amplitude of the wavefunction above the critical value—lead to formation of the black hole. Clearly, once a black hole forms, subsequent evolution in its presence will be qualitatively different from evolution in flat space. It is to be able to compute the probabilities of such events that a means of coupling quasiclassical and quantum variables is needed.

The mathematical implementation of these ideas is comparatively simple at first sight, while closer analysis reveals a number of subtleties. Consider for convenience a system consisting of one quasiclassical degree of freedom and one quantum degree of freedom. The extension to many variable systems is straightforward. In brief, one has a pair of quantum canonical variables (\check{q}, \check{p}) satisfying the canonical commutation relation $[\check{q}, \check{p}] = i\hbar$ and a commutative pair of quasiclassical canonical variables (\check{x}, \check{k}) satisfying a classical Poisson bracket relation $\{\check{x}, \check{k}\} = 1$. Analogy to the canonical commutation relations for a two-variable quantum system suggests it is natural to assume all of the canonical variables commute except \check{q}, \check{p} . This enables one to define functions of the canonical variables. The Hamiltonian is such a function, $\check{H} = H(\check{x}, \check{k}, \check{q}, \check{p}, t)$. If one forms the coupled Heisenberg-Hamilton equations using this Hamiltonian, one has the equations (at the initial time)

$$\begin{aligned} \dot{q}(t)|_{t=0} &= \frac{-i}{\hbar}[\check{q}, \check{H}], & \dot{p}(t)|_{t=0} &= \frac{-i}{\hbar}[\check{p}, \check{H}], \\ \dot{x}(t)|_{t=0} &= \{\check{x}, \check{H}\}, & \dot{k}(t)|_{t=0} &= \{\check{k}, \check{H}\}, \end{aligned} \tag{3}$$

where $q(0) = \check{q}, p(0) = \check{p}, x(0) = \check{x}, k(0) = \check{k}$.

The evolved variables $q(t), p(t), x(t)$, and $k(t)$ are in general functions of $\check{q}, \check{p}, \check{x}, \check{k}$ and t . While they divide into canonically conjugate pairs of purely quantum and purely quasiclassical type at the initial instant, once interaction begins, they generally lose their particular identification as purely quantum or quasiclassical, though they maintain their canonical conjugacy. This is a consequence of the coupling and is what enables the quasiclassical variable to come into correlation with the quantum state. Note that there will always be some combination of the evolved variables which form purely quantum and purely quasiclassical pairs, but generally not $(q(t), p(t))$ and $(x(t), k(t))$. This is the initial structure of the quasiclassical theory, and everything is fairly straightforward. The subtleties begin to appear as one looks closer.

First, the question of states must be addressed. The quantum canonical variables (\check{q}, \check{p}) are operators which act on states in a Hilbert space, as well as being algebraic elements with the canonical commutation relations. Some similar structure is needed for the quasiclassical variables to act upon. This has not been fully worked out, but the natural starting point is to treat \check{x} and \check{k} as acting on states $|(x', k'), 0\rangle$ as multiplication operators,

$$\check{x}|(x', k'), 0\rangle = x'| (x', k'), 0\rangle, \quad \check{k}|(x', k'), 0\rangle = k'| (x', k'), 0\rangle. \quad (4)$$

Despite this “operator” nature of \check{x} and \check{k} , for correspondence with familiar experience, the term operator will be reserved to functions involving the q-number operators \check{q} and \check{p} (which may involve \check{x} and \check{k} as c-number parameters). The nature of the states associated with the quasiclassical variables in the Schrödinger picture is unclear at the present time, and, in case of confusion, it is recommended that one use the Heisenberg picture where the states can be defined as ordinary joint probability distributions in (x', k') at the initial instant.

A key remark is necessary at this point about the uncertainty principle with respect to quasiclassical variables. The impression may be given by the notation that the values of both \check{x} and \check{k} are known with infinite precision. This is a false impression. As discussed above, in a real measurement situation, there is an experimental resolution, or an experimental error, to which variables are observed. The fact that a variable has been identified as quasiclassical means that its quantum fluctuations are persistently localized inside such an interval. This in turn implies that one is well above the quantum limit when observing that variable. The variable appears classical precisely

because one is not observing it too closely. In the quasiclassical approximation, one idealizes the variable as fully classical (when not interacting with quantum variables), but this is of course only a useful fiction. One cannot turn around and attempt to measure the variable more closely, or the quasiclassical approximation will break down. It is possible that it will prove useful to implement a coarse-graining on the scale of the experimental error to discourage attributing significance to fine structure in the quasiclassical variable state on scales smaller than this. The interplay between the experimental resolution and the mathematical formalism representing the quasiclassical variables is an aspect of this approach which needs further analysis.

Turn attention to the treatment of dynamics in this formalism. The first point is that the Poisson bracket is defined as

$$\{f, g\} = \frac{\partial f}{\partial \check{x}} \frac{\partial g}{\partial \check{k}} - \frac{\partial f}{\partial \check{k}} \frac{\partial g}{\partial \check{x}}. \quad (5)$$

By analogy to a two-variable classical system, it is assumed that the \check{x} and \check{k} derivatives of \check{q} and \check{p} are zero. This means that one can compute, for example,

$$\dot{x}(t)|_{t=0} = \frac{\partial H}{\partial \check{k}}.$$

This will not be a c-number if a q-number multiplies a function of \check{k} in H . The time derivative of a “classical” quantity needn’t be a c-number! This is precisely what enables the quasiclassical variables to correlate with the state of the quantum ones.

A simple example will dramatize this. Suppose that one couples a spin-1/2 particle to a quasi-classical particle through the Hamiltonian $H_I = c\check{k}\sigma_z$. The equations of motion (neglecting the self-Hamiltonian for the quasiclassical particle) are

$$\dot{x}(t) = c\sigma_z, \quad \dot{k}(t) = 0. \quad (6)$$

The solutions to the equations of motion are

$$x(t) = \check{x} + c\sigma_z t, \quad k(t) = \check{k}. \quad (7)$$

The solution for $x(t)$ involves the operator σ_z .

Suppose that the initial state of the system is given by the product state

$$|(x', 0), 0\rangle + |x\rangle, \quad (8)$$

with the spin oriented in the $+x$ direction and the particle initially at rest. The operator nature of $x(t)$ can be interpreted by decomposing the quantum state into eigenfunctions of the operator component of $x(t)$. The operator then returns a c-number eigenvalue for each component, and a probability that that eigenvalue will be realized. Here, one decomposes $|+x\rangle$ into eigenstates of σ_z and finds the evolved state in the Schrödinger picture to be

$$\frac{1}{2^{1/2}} \left(|(x' + ct, k'), t\rangle |\uparrow\rangle + |(x' - ct, k'), t\rangle |\downarrow\rangle \right). \quad (9)$$

There is a probability of $1/2$ that the quasiclassical particle will have either position $x' \pm ct$ at time t , depending on the state of the spin to which it is correlated.

As discussed above, the quasiclassical variable will have an associated experimental error. The two possible outcomes for the position of the quasiclassical particle will not be distinguishable until their centers have separated by more than this amount, and they can be resolved. After they are capable of being resolved, one has a superposition of quasiclassical (“macroscopic”) states correlated to quantum states. This is the same situation as with Schrödinger’s cat. By observing either the quasiclassical state or the spin, one destroys the superposition. One interprets the multiple possible quasiclassical outcomes as a consequence of the quantum “event” of the passage of the spin through the magnetic field implicit in the interaction Hamiltonian.

The situation in the general case is similar to this. By decomposing the quantum state into eigenfunctions of the operator part of the observable of interest, one can determine the possible values that the observable takes and with what probability. This is of course exactly the procedure one takes to predict the possible outcomes of a measurement in a fully quantum problem. If the quasiclassical state is initially in a joint probability distribution and not specified by a specific value, then one must also take this into account when determining the possible values of the quasiclassical variables in the observable.

Return to the general issue of dynamics, and consider again the equations of motion (3). These are not sufficient in themselves to determine the full evolution in general. Suppose one wanted to compute the second time derivative of \tilde{x} at $t = 0$. This should be given by the bracket of \dot{x} with the Hamiltonian, but what bracket? The first derivatives were easy to compute because they each involved a canonical variable of either purely quantum or

purely quasiclassical type. If there is nontrivial coupling between the quasiclassical and quantum variables, generally the first derivatives will be a mixture of quasiclassical and quantum variables. It is necessary to define the bracket between two such mixed expressions.

Because quantum factor ordering information is lost in the classical limit, as one canonical pair becomes quasiclassical, the quantum canonical bracket does not have a unique correspondence to a quasiclassical bracket. This is the familiar problem in the classical-quantum correspondence. There are two comparatively natural candidates for quasiclassical brackets.

One is the quasiclassical bracket proposed in [1]. For A, B functions of the quantum and quasiclassical variables,

$$\llbracket A, B \rrbracket_A = \frac{1}{i\hbar}[A, B] + \left(\frac{\partial A}{\partial \tilde{x}} \frac{\partial B}{\partial \tilde{k}} - \frac{\partial A}{\partial \tilde{k}} \frac{\partial B}{\partial \tilde{x}} \right). \quad (10)$$

If $A = Uf$ and $B = Vg$, where U, V are functions of \tilde{q}, \tilde{p} and f, g are functions of \tilde{x}, \tilde{k} , this takes the form

$$\llbracket Uf, Vg \rrbracket_A = \frac{1}{i\hbar}[U, V]fg + i\hbar UV\{f, g\}. \quad (11)$$

This bracket is not antisymmetric and hence not hermitian.

A second bracket, which is antisymmetric and hermitian, is the bracket proposed independently by Alexandrov[6] and by Boucher and Traschen[7] (ABT). For A, B functions of the quantum and quasiclassical variables,

$$\llbracket A, B \rrbracket = \frac{1}{i\hbar}[A, B] + \frac{1}{2} \left(\frac{\partial A}{\partial \tilde{x}} \frac{\partial B}{\partial \tilde{k}} - \frac{\partial A}{\partial \tilde{k}} \frac{\partial B}{\partial \tilde{x}} + \frac{\partial B}{\partial \tilde{k}} \frac{\partial A}{\partial \tilde{x}} - \frac{\partial B}{\partial \tilde{x}} \frac{\partial A}{\partial \tilde{k}} \right). \quad (12)$$

If $A = Uf$ and $B = Vg$, this is

$$\llbracket Uf, Vg \rrbracket = \frac{1}{i\hbar}[U, V]fg + \frac{1}{2}(UV + VU)\{f, g\}. \quad (13)$$

Both of these brackets give the correct relations among the canonical variables (\tilde{q}, \tilde{p}) and (\tilde{x}, \tilde{k}) , but note that the factor of $i\hbar$ has been divided out of the purely quantum commutator. Both can be obtained by taking the classical limit in an appropriate way[8]. Choosing one imposes a canonical structure on the algebra of functions of all the canonical variables.

An important issue is whether these brackets are derivations, that is, whether they satisfy a product rule[9],

$$[[A, BC]] = [[A, B]]C + B[[A, C]]. \quad (14)$$

The answer is that neither is unconditionally a derivation[8]. The problem is that taking the bracket of a variable is like taking a time derivative, and as we have already seen, taking a time derivative can change a c-number into something q-number valued. The result is that the factors in a product which commute initially may not commute with the factors produced by taking a derivative or bracket. Since the outcome depends on the order of factors, a product rule will not hold in general. A preferred ordering must hold initially to have a product rule. By choosing such an ordering, one is not affecting the value of the bracket, only making it possible to evaluate with a product rule.

To be precise, consider the quasiclassical bracket (10). For A, B, C general functions of quantum and quasiclassical variables, one finds[8]

$$[[A, BC]]_A = [[A, B]]_A C + B[[A, C]]_A + \left[\frac{\partial A}{\partial \check{x}}, B\right] \frac{\partial C}{\partial \check{k}} - \left[\frac{\partial A}{\partial \check{k}}, B\right] \frac{\partial C}{\partial \check{x}}. \quad (15)$$

Since this bracket is not antisymmetric, there is a different rule when acting from the right

$$[[BC, A]]_A = [[B, A]]_A C + B[[C, A]]_A + \frac{\partial B}{\partial \check{x}} \left[C, \frac{\partial A}{\partial \check{k}}\right] - \frac{\partial B}{\partial \check{k}} \left[C, \frac{\partial A}{\partial \check{x}}\right]. \quad (16)$$

If one decomposes BC as a sum of terms of the form fU with f on the left, where U is quantum and f is quasiclassical, then a product rule holds in the first case. In the second case, a product rule holds if BC is decomposed as a sum of terms Uf , with f on the right.

A similar result holds for the ABT bracket (12). There, one finds

$$[[A, BC]] = -[[BC, A]] = [[A, B]]C + B[[A, C]] + \frac{1}{2} \left(\left[\frac{\partial A}{\partial \check{x}}, B\right] \frac{\partial C}{\partial \check{k}} - \left[\frac{\partial A}{\partial \check{k}}, B\right] \frac{\partial C}{\partial \check{x}} + \frac{\partial B}{\partial \check{k}} \left[C, \frac{\partial A}{\partial \check{x}}\right] - \frac{\partial B}{\partial \check{x}} \left[C, \frac{\partial A}{\partial \check{k}}\right] \right). \quad (17)$$

If one decomposes BC as a sum of symmetrically ordered terms $\frac{1}{2}(fU + Uf)$, where U is quantum and f is quasiclassical, then a product rule holds,

$$[[A, \frac{1}{2}(fU + Uf)]] = \frac{1}{2} \left([[A, f]]U + f[[A, U]] + [[A, U]]f + U[[A, f]] \right). \quad (18)$$

Because of the particular ordering of the quantum operators U, V in (11), the bracket is seen not to be antisymmetric and hence not hermitian. This leads to the possibility that $\llbracket H, H \rrbracket_A \neq 0$, which in turn can lead to the peculiar situation that an ostensibly time-independent Hamiltonian has a time-dependent evolution. These features seriously complicate evolution and may be unphysical, so this bracket will *not* be used. The ABT quasiclassical bracket (12) is antisymmetric and hermitian and will be used for evolution.

Having chosen the quasiclassical bracket, one can now formulate the derivative of a general time-dependent function. The equation of motion for a function $A(q(t), p(t), x(t), k(t), t)$ with initial value $A(\check{q}, \check{p}, \check{x}, \check{k}, 0)$ is

$$\begin{aligned} \frac{dA(q(t), p(t), x(t), k(t), t)}{dt} = & \llbracket A(q(t), p(t), x(t), k(t), t), \check{H} \rrbracket \\ & + \frac{\partial A(q(t), p(t), x(t), k(t), t)}{\partial t}, \end{aligned} \quad (19)$$

where $\check{H} = H(\check{q}, \check{p}, \check{x}, \check{k}, t)$ is the Hamiltonian in terms of the initial variables. In particular, this gives the equations of motion for $q(t), p(t), x(t)$, and $k(t)$,

$$\begin{aligned} \dot{q}(t) &= \llbracket q(t), \check{H} \rrbracket, & \dot{p}(t) &= \llbracket p(t), \check{H} \rrbracket, \\ \dot{x}(t) &= \llbracket x(t), \check{H} \rrbracket, & \dot{k}(t) &= \llbracket k(t), \check{H} \rrbracket. \end{aligned} \quad (20)$$

It is very important to emphasize that $\check{H} = H(\check{q}, \check{p}, \check{x}, \check{k}, t)$ is the Hamiltonian expressed in terms of the initial variables. This is necessary to be able to evaluate the bracket. If A were expressed in terms of the original variables, one could use (12) to evaluate the bracket. Alternatively, one could put H into symmetrically ordered form and use the product rule (18) to simplify the bracket. The ordering rule which enables the bracket to satisfy a product rule is only known in terms of the initial variables. This is because the multiplicative properties of the canonical variables can change with time, so that one may have $[x(t), k(t)] \neq 0$. The requirement that an expression be symmetrically ordered as a product of a c-number and a q-number cannot be easily satisfied in terms of the evolved variables.

Furthermore, $x(t)$ and $k(t)$ are not generally c-numbers, even if they happen to mutually commute. One cannot take derivatives with respect to them (without extending the definition of the derivative). This means particularly that the quasiclassical bracket is not given in terms of the evolved variables by an expression of the form (12) with \check{x}, \check{k} replaced by $x(t), k(t)$.

Nevertheless, one desires that the canonical relations between the canonical variables computed with the quasiclassical bracket be preserved in time, e.g. $\llbracket x(t), k(t) \rrbracket = 1$. In purely quantum or classical theory, this follows from the Jacobi identity for the bracket, but the Jacobi identity does not hold in general for the quasiclassical bracket[11]. One finds

$$\begin{aligned} \llbracket \llbracket A, B \rrbracket, C \rrbracket - \llbracket \llbracket A, C \rrbracket, B \rrbracket - \llbracket A, \llbracket B, C \rrbracket \rrbracket &= \\ &= \{\{A, B\}, C\} - \{A, \{B, C\}\} - \{\{A, C\}, B\} + \{A, \{C, B\}\} \\ &\quad \{\{B, C\}, A\} - \{B, \{C, A\}\} - \{\{B, A\}, C\} + \{B, \{A, C\}\} \\ &\quad \{\{C, A\}, B\} - \{C, \{A, B\}\} - \{\{C, B\}, A\} + \{C, \{B, A\}\}. \end{aligned} \quad (21)$$

The right hand side of this equation would vanish if the Jacobi identity were satisfied. The main difficulty is the noncommutative nature of A, B, C , but accepting the ordering it becomes as if one is missing part of the Jacobi identity as it applies to the Poisson bracket. There are obvious additional terms that one could add (maintaining ordering) which would cause this to vanish, but there does not seem to be a way to redefine the bracket so that they occur naturally. For instance, a term like $-\{\{A, C\}, B\}$ but where the differentiated B is ordered between A and C would cancel against the first two terms on the right hand side.

For some Hamiltonians having special forms (particularly not coupling both the coordinates and momenta of quasiclassical and quantum variables), a special case of the Jacobi identity holds and it is sufficient to preserve the brackets of the fundamental canonical variables. One might conclude that the quasiclassical approximation is not a good one for Hamiltonians not of one of these forms. Naturally one hopes that physically interesting Hamiltonians are consistent, but this has not been proven and may not be true. Work is in progress to clarify this important issue.

The fact, $\llbracket H, H \rrbracket = 0$, implies that the only time-dependence H has is its explicit dependence. This is good because it means that

$$H(q(t), p(t), x(t), k(t), t) = H(\check{q}, \check{p}, \check{x}, \check{k}, t) = \check{H},$$

even though the detailed expression of H in terms of the evolved variables may have an ordering which is not immediately obvious. The equation (20) is not in fact different from what one would naively expect.

The inconvenience of having to work with the initial variables is not as serious as one might imagine. When solving the Heisenberg equations of

motion in quantum theory, one is trying to find the expression for the evolved variables in terms of the initial ones. Having found a candidate solution, the equations are verified by computing the commutator in the initial variables. It is the same here.

Solutions to the equations of motion (20) are most easily found by developing a Taylor series expansion in time about the initial value. This is done by evaluating higher time derivatives at the initial time by taking further commutators with H . Since everything is evaluated at the initial time, one can proceed iteratively with little difficulty using (12) to evaluate the bracket expressions. A second solution technique would be to use canonical transformations[1, 10], but further work on this is needed.

The goal of the quasiclassical approach is to approximate a fully quantum theory by treating approximately classical degrees of freedom as classical when they are present in isolation yet coupling them to the quantum variables in such a way that they may come into correlation with the quantum state during interaction. The possibility of correlation between a quasiclassical variable and the states in a quantum superposition is the essential feature captured in this approach which is both observed physically and yet is absent from the traditional semiclassical description of coupling to the expectation value. The quasiclassical approximation is implemented by neglecting the self-interference effects of degrees of freedom which are persistently localized within their experimental uncertainty.

A candidate mathematical approach to the quasiclassical approximation treats the canonical conjugates associated to the quasiclassical degrees of freedom as multiplicatively commutative and retains their canonical conjugacy through a classical Poisson bracket. This makes these degrees of freedom behave classically in isolation. The coupling to quantum degrees of freedom is accomplished by considering functions of both commutative and noncommutative variables. A quasiclassical bracket is defined which preserves the canonical structure of the classical and quantum subalgebras and extends it to pairs of functions of the mixed set of variables. This bracket is antisymmetric and hermitian and can be used to define equations of motion which are essentially coupled Hamilton-Heisenberg equations. The complications are that the candidate quasiclassical bracket satisfies a product rule only when acting on quantities ordered in a particular way and the Jacobi identity does not hold generally. As a consequence, it is not certain how much of the canonical structure is preserved under evolution. The canonical relations

among the fundamental canonical variables are preserved for special Hamiltonians, and work is in progress to determine for what class of Hamiltonians this is true.

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References

- [1] A. Anderson, Phys. Rev. Lett. **74**, 621 (1995).
- [2] T. Kibble, in *Quantum Gravity 2*, edited by C.J. Isham, R. Penrose, and D.W. Sciama, (Oxford Univ. Press, 1981), p. 63.
- [3] The measurements may be thought of as being repeated in time, but this is a separate kind of repetition already taken into account by the changing of the expectation value with time. The repetition involved in computing the expectation value refers to a collection of independent measurements made on identically prepared systems at the same point in their evolution.
- [4] A. Guth and S.Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); S. Hawking, Phys. Lett. B **115**, 295 (1982); A. Starobinskii, Phys. Lett. B **117**, 175 (1982).
- [5] M. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
- [6] I.V. Aleksandrov, Z. Naturf. **36A**, 902 (1981).
- [7] W. Boucher and J. Traschen, Phys. Rev. **D37**, 3522 (1988).
- [8] A. Anderson, “Reply to Comment on ‘Quantum backreaction on classical variables’ ”, submitted to Phys. Rev. Lett. (1995).
- [9] L. Diósi, “Comment on ‘Quantum backreaction on classical variables’ ”, submitted to Phys. Rev. Lett. (1995).
- [10] A. Anderson, Ann. Phys. (NY) **232**, 292 (1994); Phys. Lett. B **319**, 157 (1993).
- [11] A. Anderson, in preparation (1995).